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NON-NEWTONIAN MERSENNE AND MERSENNE LUCAS SEQUENCES

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Abstract

In this paper we introduce a new class of Mersenne and Mersenne-Lucas numbers, constructed in the framework of non-Newtonian calculus, that we call respectively, non-Newtonian Mersenne and non-Newtonian Mersenne-Lucas numbers. From the view point of non-Newtonian calculus, we investigate some important identities and formulas of the classical Mersenne numbers and Mersenne-Lucas numbers. As a result we prove several relations of non-Newtonian Mersenne and Mersenne-Lucas numbers. We also study some other properties of these non-Newtonian numbers, such as Catalan-like identities, Cassini-like identities, Binet-like formulas and generating functions.

1. Introduction

The initial endeavour in non-Newtonian calculus originated with Grossman and Katz in 1972. They modified the calculus developed by Newton and Leibniz in the 1700s, since a recognised method for developing a familiar mathematical system involves altering the assumptions of an existing one. Additionally, they presented an infinite family of calculus that encompasses specialised calculi, including anageometric calculus, bigeometric calculus, harmonic calculus, and geometric calculus. The addition, subtraction, multiplication, division, ordering, derivative, and integral operations in each of these new calculi vary, which leads to novel applications for intriguing mathematical and related problems, some of which are in quantum calculus, functional analysis, complex analysis, fractal geometry, differential equations, calculus of variations, image analysis, signal processing, and economics.

The Fibonacci and Lucas numbers have been generalised in a different method by De Ágirmen and Duyar [9]. The authors created a new form of Fibonacci and Lucas numbers known as non-Newtonian Fibonacci and non-Newtonian Lucas numbers by combining the Fibonacci numbers, Lucas numbers, and non-Newtonian real numbers. They also investigated a few characteristics of these numbers.

The concepts discussed in [9] are extended to Mersenne and Mersenne-Lucas numbers in this study. This paper's primary goal is to present and examine non-Newtonian Mersenne and non-Newtonian Mersenne-Lucas numbers. This requires some prior knowledge of Mersenne and Mersenne-Lucas numbers[5-8, 10 as well as Non-Newtonian calculus[1-4].

Although the term "arithmetic" is typically used to refer to positive integers, in this context it refers to an integer ordered field whose universe is a subset of the set of real numbers. Let $\alpha: R \rightarrow A \subseteq R$ be a bijection. Then, it is called a generator and defines an arithmetic. A non-Newtonian real number is any element in the range of α, represented by R_{α}. The classical arithmetic is obtained choosing $\alpha = I$, and it results $R_I = R$.

$$\alpha - arithmetic$$

$Realm$	R_{α}
$\alpha - addition$	$\rho \dot{+} \tau = \alpha\{\alpha^{-1}(\rho) + \alpha^{-1}(\tau)\}$
$\alpha - subtraction$	$\rho \dot{-} \tau = \alpha\{\alpha^{-1}(\rho) - \alpha^{-1}(\tau)\}$
$\alpha - multiplication$	$\rho \dot{\times} \tau = \alpha\{\alpha^{-1}(\rho) \times \alpha^{-1}(\tau)\}$
$\alpha - division$	$\rho \dot{/} \tau = \frac{\rho}{\tau}\alpha = \alpha\left\{\frac{\alpha^{-1}(\rho)}{\alpha^{-1}(\tau)}\right\} (\tau \neq 0)$
$\alpha - ordering$	$\rho \dot{<} \tau \Leftrightarrow \alpha^{-1}(\rho) < \alpha^{-1}(\tau)$

where $\rho, \tau \in R_{\alpha}$.

A number of the type $M_n = 2^n - 1$, where n is an integer, was first developed in 1644 by the French mathematician Marin Mersenne. The Mersenne sequences have been investigated as part of numerous studies. The definition of Mersenne-Lucas sequences is $ML_n = 2^n + 1, n \geq 2$, with $ML_0 = 2, ML_1 = 3$.

2. Main Results:

Definition: 2.1

Non-Newtonian Mersenne and Non-Newtonian Mersenne Lucas sequences are defined as $N\mathfrak{M}_n = \alpha(\mathfrak{M}_n)$ and $N\mathfrak{M}\mathrm{L}_n = \alpha(\mathfrak{M}\mathrm{L}_n)$ where $\mathfrak{M}_n$ and $\mathfrak{M}\mathrm{L}_n$ are n^{th} Mersenne and Mersenne Lucas sequences.

The set of Non-Newtonian Mersenne and Non-Newtonian Mersenne Lucas sequences which we denote by $N\mathfrak{M}_n$ and $N\mathfrak{M}\mathrm{L}_n$ are as follows:

$$N\mathfrak{M} = \{N\mathfrak{M}_n : n \in \mathbb{N}\}$$

$$= \{\dot{0}, \dot{1}, \dot{3}, \dot{7}, \dot{15}, \dot{31}, \dots \alpha(\mathfrak{M}_n), \dots\}$$

and
$$N\mathfrak{M}\mathrm{L} = \{\mathrm{N}\mathfrak{M}\mathrm{L}_n : n \in \mathbb{N}\}$$

$$= \{\dot{2}, \dot{3}, \dot{5}, \dot{9}, \dot{17}, \dot{33}, \dots \alpha(\mathfrak{M}\mathrm{L}_n), \dots\}$$

Choosing the generator I defined by $\alpha(y) = y$ for every $y \in \mathbb{R}$, we obtain Mersenne and Mersenne-Lucas numbers with respect to classical arithmetic, respectively.

Also , if we consider the generator exp defined by $\alpha(y) = e^y$ for each $y \in \mathbb{R}$, we obtain Mersenne and Mersenne-Lucas numbers with respect to geometric arithmetic, respectively, as follows:

$$NG\mathfrak{M} = \{\alpha(\mathfrak{M}_n) : n \in \mathbb{N}\}$$

$$= \{e^{\mathfrak{M}_n} : n \in \mathbb{N}\}$$

$$= \{e^0, e^1, e^3, e^7, e^{15}, e^{31}, \dots e^{\mathfrak{M}_n}, \dots\}$$

and

$$NG\mathfrak{M}\mathrm{L} = \{\alpha(\mathfrak{M}\mathrm{L}_n) : n \in \mathbb{N}\}$$

$$= \{e^{\mathfrak{M}\mathrm{L}_n} : n \in \mathbb{N}\}$$

$$= \{e^2, e^3, e^5, e^9, e^{17}, e^{33}, \dots e^{\mathfrak{M}\mathrm{L}_n}, \dots\}$$

Theorem: 2.1

(i) $N\mathfrak{M}_{n+2} = \dot{3} \dot{\times} N\mathfrak{M}_{n+1} \dot{-} \dot{2} \dot{\times} N\mathfrak{M}_n$

(ii) $N\mathfrak{M}\mathrm{L}_{n+2} = \dot{3} \dot{\times} N\mathfrak{M}\mathrm{L}_{n+1} \dot{-} \dot{2} \dot{\times} N\mathfrak{M}\mathrm{L}_n$

Proof:

(i) $\dot{3} \dot{\times} N\mathfrak{M}_{n+1} \dot{-} \dot{2} \dot{\times} N\mathfrak{M}_n = \alpha(3) \dot{\times} \alpha(\mathfrak{M}_{n+1}) \dot{-} \alpha(2) \dot{\times} \alpha(\mathfrak{M}_n)$

$$= \alpha\{\alpha^{-1}\alpha(\alpha^{-1}\alpha(3) \times \alpha^{-1}\alpha(\mathfrak{M}_{n+1})) - \alpha^{-1}\alpha(\alpha^{-1}\alpha(2) \times \alpha^{-1}\alpha(\mathfrak{M}_n))\}$$

$$= \alpha(3\mathfrak{M}_{n+1} - 2\mathfrak{M}_n)$$

$$= \alpha(\mathfrak{M}_{n+2})$$

$$= N\mathfrak{M}_{n+2}$$

(ii) $\dot{3} \dot{\times} N\mathfrak{M}\mathrm{L}_{n+1} \dot{-} \dot{2} \dot{\times} N\mathfrak{M}\mathrm{L}_n = \alpha(3) \dot{\times} \alpha(\mathfrak{M}\mathrm{L}_{n+1}) \dot{-} \alpha(2) \dot{\times} \alpha(\mathfrak{M}\mathrm{L}_n)$

$$= \alpha\{\alpha^{-1}\alpha(\alpha^{-1}\alpha(3) \times \alpha^{-1}\alpha(\mathfrak{M}\mathrm{L}_{n+1})) - \alpha^{-1}\alpha(\alpha^{-1}\alpha(2) \times \alpha^{-1}\alpha(\mathfrak{M}\mathrm{L}_n))\}$$

$$= \alpha(3\mathfrak{M}\mathrm{L}_{n+1} - 2\mathfrak{M}\mathrm{L}_n)$$

$$= \alpha(\mathfrak{M}\mathrm{L}_{n+2})$$

$$= N\mathfrak{M}\mathrm{L}_{n+2}$$

Theorem: 2.2

The generating functions for Non-Newtonian Mersenne and Mersenne-Lucas sequences are given by

(i) $g(s) = \frac{s}{\dot{1} \dot{-} \dot{3} \dot{\times} s \dot{+} \dot{2} \dot{\times} s^{\dot{2}}} \alpha$

(ii) $h(s) = \frac{\dot{2} \dot{-} \dot{3} \dot{\times} s}{\dot{1} \dot{-} \dot{3} \dot{\times} s \dot{+} \dot{2} \dot{\times} s^{\dot{2}}} \alpha$

Proof:

(i) $g(s) = \alpha \sum_{n=0}^{\infty} N\mathfrak{M}_n \dot{\times} s^{\dot{n}}$

$$= s \dot{+} \alpha \sum_{n=2}^{\infty} N\mathfrak{M}_n \dot{\times} s^{\dot{n}}$$

$$= s \dot{+} \alpha \sum_{n=2}^{\infty} \left(\dot{3} \dot{\times} N\mathfrak{M}_{n-1} \dot{-} \dot{2} \dot{\times} N\mathfrak{M}_{n-2}\right) \dot{\times} s^{\dot{n}}$$

$$= s \dot{+} \alpha \sum_{n=2}^{\infty} \left(\dot{3} \dot{\times} N\mathfrak{M}_{n-1}\right) \dot{\times} s^{\dot{n}} \dot{+} \alpha \sum_{n=2}^{\infty} \left(\dot{-} \dot{2} \dot{\times} N\mathfrak{M}_{n-2}\right) \dot{\times} s^{\dot{n}}$$

$$= s \dot{+} \dot{3} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-1}\right) \dot{\times} s^{\dot{n}} \dot{-} \dot{2} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-2}\right) \dot{\times} s^{\dot{n}}$$

$$s \dot{\times} g(s) = \alpha \sum_{n=0}^{\infty} N\mathfrak{M}_n \dot{\times} s^{\dot{n} \dot{+} \dot{1}} = \alpha \sum_{n=2}^{\infty} N\mathfrak{M}_{n-1} \dot{\times} s^{\dot{n}}$$

$$s^{\dot{2}} \dot{\times} g(s) = \alpha \sum_{n=0}^{\infty} N\mathfrak{M}_n \dot{\times} s^{\dot{n} \dot{+} \dot{2}} = \alpha \sum_{n=2}^{\infty} N\mathfrak{M}_{n-2} \dot{\times} s^{\dot{n}}$$

$$\left(\dot{1} \dot{-} \dot{3} \dot{\times} s \dot{+} \dot{2} \dot{\times} s^{\dot{2}}\right) \dot{\times} g(s) = g(s) \dot{-} \dot{3} \dot{\times} \left(s \dot{\times} g(s)\right) \dot{+} \dot{2} \dot{\times} \left(s^{\dot{2}} \dot{\times} g(s)\right)$$

$$= s \dot{+} \dot{3} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-1} \dot{\times} s^{\dot{n}}\right) \dot{-} \dot{2} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-2} \dot{\times} s^{\dot{n}}\right)$$

$$\dot{-} \dot{3} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-1} \dot{\times} s^{\dot{n}}\right) \dot{+} \dot{2} \dot{\times} \alpha \sum_{n=2}^{\infty} \left(N\mathfrak{M}_{n-2} \dot{\times} s^{\dot{n}}\right)$$

$$= s.$$

Hence, $g(s) = \frac{s}{\dot{1} \dot{-} \dot{3} \dot{\times} s \dot{+} \dot{2} \dot{\times} s^{\dot{2}}} \alpha$

(ii) $h(s) = \alpha \sum_{n=0}^{\infty} N\mathfrak{M}\mathrm{L}_n \dot{\times} s^{\dot{n}}$

$$= \dot{2}\dot{+}\dot{3}\dot{\times}s\dot{+}\alpha\sum_{n=2}^{\infty}N\mathfrak{M}\mathrm{L}_n\dot{\times}s^{\dot{n}}$$

$$= \dot{2}\dot{+}\dot{3}\dot{\times}s\dot{+}\alpha\sum_{n=2}^{\infty}\left(\dot{3}\dot{\times}N\mathfrak{M}\mathrm{L}_{n-1}\dot{-}\dot{2}\dot{\times}N\mathfrak{M}\mathrm{L}_{n-2}\right)\dot{\times}s^{\dot{n}}$$

$$= \dot{2}\dot{+}\dot{3}\dot{\times}s\dot{+}\alpha\sum_{n=2}^{\infty}\left(\dot{3}\dot{\times}N\mathfrak{M}\mathrm{L}_{n-1}\right)\dot{\times}s^{\dot{n}}\dot{+}\alpha\sum_{n=2}^{\infty}\left(\dot{-}\dot{2}\dot{\times}N\mathfrak{M}\mathrm{L}_{n-2}\right)\dot{\times}s^{\dot{n}}$$

$$= \dot{2}\dot{+}\dot{3}\dot{\times}s\dot{+}\dot{3}\dot{\times}\alpha\sum_{n=2}^{\infty}\left(N\mathfrak{M}\mathrm{L}_{n-1}\right)\dot{\times}s^{\dot{n}}\dot{-}\dot{2}\dot{\times}\alpha\sum_{n=2}^{\infty}\left(N\mathfrak{M}\mathrm{L}_{n-2}\right)\dot{\times}s^{\dot{n}}$$

$$s\dot{\times}h(s) = \alpha\sum_{n=0}^{\infty}N\mathfrak{M}\mathrm{L}_n\dot{\times}s^{\dot{n}\dot{+}1} = \dot{2}\dot{\times}s\dot{+}\alpha\sum_{n=2}^{\infty}N\mathfrak{M}\mathrm{L}_{n-1}\dot{\times}s^{\dot{n}}$$

$$s^{\dot{2}}\dot{\times}h(s) = \alpha\sum_{n=0}^{\infty}N\mathfrak{M}\mathrm{L}_n\dot{\times}s^{\dot{n}\dot{+}\dot{2}} = \alpha\sum_{n=2}^{\infty}N\mathfrak{M}\mathrm{L}_{n-2}\dot{\times}s^{\dot{n}}$$

$$\left(\dot{1}\dot{-}\dot{3}\dot{\times}s\dot{+}\dot{2}\dot{\times}\dot{s}^{2}\right)\dot{\times}h(s) = h(s)\dot{-}\dot{3}\dot{\times}\left(s\dot{\times}h(s)\right)\dot{+}\dot{2}\dot{\times}\left(s^{\dot{2}}\dot{\times}h(s)\right)$$

$$= \dot{2}\dot{+}\dot{3}\dot{\times}s\dot{+}\dot{3}\dot{\times}\alpha\sum_{n=2}^{\infty}\left(N\mathfrak{M}\mathrm{L}_{n-1}\dot{\times}s^{\dot{n}}\right)\dot{-}\dot{2}\dot{\times}\alpha\sum_{n=2}^{\infty}\left(N\mathfrak{M}\mathrm{L}_{n-2}\dot{\times}s^{\dot{n}}\right)$$

$$\dot{-}\dot{3}\dot{\times}\left(\dot{2}\dot{\times}s\dot{+}\alpha\sum_{n=2}^{\infty}\left(N\mathfrak{M}\mathrm{L}_{n-1}\dot{\times}s^{\dot{n}}\right)\right)\dot{+}\dot{2}\dot{\times}\left(\alpha\sum_{n=2}^{\infty}N\mathfrak{M}\mathrm{L}_{n-2}\dot{\times}s^{\dot{n}}\right)$$

$$= \dot{2}\dot{-}\dot{3}\dot{\times}s$$

Hence, $h(s) = \frac{\dot{2}\dot{-}3\dot{\times}s}{\dot{1}\dot{-}\dot{3}\dot{\times}s\dot{+}\dot{2}\dot{\times}s^{\dot{2}}}\alpha$

Theorem: 2.3

The Binet formula for for Non-Newtonian Mersenne and Mersenne-Lucas sequences are given by

(i) $N\mathfrak{M}_n = \dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}}$

(ii) $N\mathfrak{M}\mathrm{L}_n = \dot{r_1}^{\dot{n}}\dot{+}\dot{r_2}^{\dot{n}}$ where $\dot{r_1} = \dot{1}, \dot{r_2} = \dot{2}$

Proof:

(i) $\dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}} = (\dot{r_1}\dot{\times}\dot{r_1}\dot{\times}\dot{r_1}\dot{\times}\ldots(n\ times))\dot{-}(\dot{r_2}\dot{\times}\dot{r_2}\dot{\times}\dot{r_2}\dot{\times}\ldots(n\ times))$

$$= \alpha(r_1^n)\dot{-}\alpha(r_2^n) = \alpha\left(\alpha^{-1}\alpha(r_1^n) - \alpha^{-1}\alpha(r_2^n)\right)$$

$$= \alpha(r_1^n - r_2^n) = \alpha(\mathfrak{M}_n) = N\mathfrak{M}_n$$

(ii) $\dot{r_1}^{\dot{n}} \dot{+} \dot{r_2}^{\dot{n}} = (\dot{r_1} \dot{\times} \dot{r_1} \dot{\times} \dot{r_1} \dot{\times} \dots (n\ times)) \dot{+} (\dot{r_2} \dot{\times} \dot{r_2} \dot{\times} \dot{r_2} \dot{\times} \dots (n\ times))$

$$= \alpha(r_1^n) \dot{+} \alpha(r_2^n) = \alpha\left(\alpha^{-1}\alpha(r_1^n) + \alpha^{-1}\alpha(r_2^n)\right)$$
$$= \alpha(r_1^n + r_2^n) = \alpha(\mathfrak{M}L_n) = N\mathfrak{M}L_n$$

Theorem: 2.4 (Vajda Identity)

Let n, rand s be positive integers. Then, we have

(i) $N\mathfrak{M}_{n+r} \dot{\times} N\mathfrak{M}_{n+s} \dot{-} N\mathfrak{M}_n \dot{\times} N\mathfrak{M}_{n+r+s} = \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{M}_r \dot{\times} N\mathfrak{M}_s$

(ii) $N\mathfrak{M}L_{n+r} \dot{\times} N\mathfrak{M}L_{n+s} \dot{-} N\mathfrak{M}L_n \dot{\times} N\mathfrak{M}L_{n+r+s} = \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{M}_r \dot{\times} N\mathfrak{M}_s$

Proof:

(i) $N\mathfrak{M}_{n+r} \dot{\times} N\mathfrak{M}_{n+s} \dot{-} N\mathfrak{M}_n \dot{\times} N\mathfrak{M}_{n+r+s} = \alpha(\mathfrak{M}_{n+r}) \dot{\times} \alpha(\mathfrak{M}_{n+s}) \dot{-} \alpha(\mathfrak{M}_n) \dot{\times} \alpha(\mathfrak{M}_{n+r+s})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n+r}) \times \alpha(\mathfrak{M}_{n+s})\right)\right) \dot{-} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_n) \times \alpha(\mathfrak{M}_{n+r+s})\right)\right)$$

$$= \alpha(\mathfrak{M}_{n+r}\mathfrak{M}_{n+s}) \dot{-} \alpha(\mathfrak{M}_n\mathfrak{M}_{n+r+s})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n+r}\mathfrak{M}_{n+s}) - \alpha(\mathfrak{M}_n\mathfrak{M}_{n+r+s})\right)\right)$$

$$= \alpha(\mathfrak{M}_{n+r}\mathfrak{M}_{n+s} - \mathfrak{M}_n\mathfrak{M}_{n+r+s})$$

$$= \alpha(2^n\mathfrak{M}_r\mathfrak{M}_s) = \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{M}_r \dot{\times} N\mathfrak{M}_s$$

(ii) $N\mathfrak{M}L_{n+r} \dot{\times} N\mathfrak{M}L_{n+s} \dot{-} N\mathfrak{M}L_n \dot{\times} N\mathfrak{M}L_{n+r+s} =$
$\alpha(\mathfrak{M}L_{n+r}) \dot{\times} \alpha(\mathfrak{M}L_{n+s}) \dot{-} \alpha(\mathfrak{M}L_n) \dot{\times} \alpha(\mathfrak{M}L_{n+r+s})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n+r}) \times \alpha(\mathfrak{M}L_{n+s})\right)\right) \dot{-} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_n) \times \alpha(\mathfrak{M}L_{n+r+s})\right)\right)$$

$$= \alpha(\mathfrak{M}L_{n+r}\mathfrak{M}L_{n+s}) \dot{-} \alpha(\mathfrak{M}L_n\mathfrak{M}L_{n+r+s})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n+r}\mathfrak{M}L_{n+s}) - \alpha(\mathfrak{M}L_n\mathfrak{M}L_{n+r+s})\right)\right)$$

$$= \alpha(\mathfrak{M}L_{n+r}\mathfrak{M}L_{n+s} - \mathfrak{M}L_n\mathfrak{M}L_{n+r+s}) = \alpha(-2^n\mathfrak{M}L_r\mathfrak{M}L_s)$$

$$= \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{M}_r \dot{\times} N\mathfrak{M}_s$$

Theorem 2.5 (Catalan's identity)

(i) $N\mathfrak{M}_{n-s} \dot{\times} N\mathfrak{M}_{n+s} \dot{-} N\mathfrak{M}_{n}^{\dot{2}} = \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}\dot{-}s} \dot{\times} N\mathfrak{M}_{s}{}^{2}$

(ii) $N\mathfrak{M}L_{n-s} \dot{\times} N\mathfrak{M}L_{n+s} \dot{-} N\mathfrak{M}L_{n}^{\dot{2}} = \dot{2}^{\dot{n}\dot{-}s} \dot{\times} N\mathfrak{M}_{s}{}^{2}$

Proof:

(i)
$$\begin{aligned} N\mathfrak{M}_{n-s} \dot{\times} N\mathfrak{M}_{n+s} \dot{-} N\mathfrak{M}_{n}^{\dot{2}} &= \alpha(\mathfrak{M}_{n-s}) \dot{\times} \alpha(\mathfrak{M}_{n+s}) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n-s}) \times \alpha(\mathfrak{M}_{n+s})\right)\right) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha(\mathfrak{M}_{n-s}\mathfrak{M}_{n+s}) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n-s}\mathfrak{M}_{n+s}) - \alpha(\mathfrak{M}_{n}^{2})\right)\right) \\ &= \alpha(\mathfrak{M}_{n-s}\mathfrak{M}_{n+s} - \mathfrak{M}_{n}^{2}) \\ &= \alpha(-2^{n-s}\mathfrak{M}_{s}^{2}) = \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}\dot{-}s} \dot{\times} N\mathfrak{M}_{s}{}^{2} \end{aligned}$$

(ii)
$$\begin{aligned} N\mathfrak{M}L_{n-s} \dot{\times} N\mathfrak{M}L_{n+s} \dot{-} N\mathfrak{M}L_{n}^{\dot{2}} &= \alpha(\mathfrak{M}L_{n-s}) \dot{\times} \alpha(\mathfrak{M}L_{n+s}) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n-s}) \times \alpha(\mathfrak{M}L_{n+s})\right)\right) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \\ &= \alpha(\mathfrak{M}L_{n-s}\mathfrak{M}L_{n+s}) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n-s}\mathfrak{M}L_{n+s}) - \alpha(\mathfrak{M}L_{n}^{2})\right)\right) \\ &= \alpha(\mathfrak{M}L_{n-s}\mathfrak{M}L_{n+s} - \mathfrak{M}L_{n}^{2}) \\ &= \alpha(2^{n-s}\mathfrak{M}_{s}^{2}) = \dot{2}^{\dot{n}\dot{-}s} \dot{\times} N\mathfrak{M}_{s}{}^{2} \end{aligned}$$

Theorem 2.6 (Cassini's identity)

(i) $N\mathfrak{M}_{n-1} \dot{\times} N\mathfrak{M}_{n+1} \dot{-} N\mathfrak{M}_{n}^{\dot{2}} = \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}\dot{-}1}$

(ii) $N\mathfrak{M}L_{n-1} \dot{\times} N\mathfrak{M}L_{n+1} \dot{-} N\mathfrak{M}L_{n}^{\dot{2}} = \dot{2}^{\dot{n}\dot{-}1}$

Proof:

(i)
$$\begin{aligned} N\mathfrak{M}_{n-1} \dot{\times} N\mathfrak{M}_{n+1} \dot{-} N\mathfrak{M}_{n}^{\dot{2}} &= \alpha(\mathfrak{M}_{n-1}) \dot{\times} \alpha(\mathfrak{M}_{n+1}) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n-1}) \times \alpha(\mathfrak{M}_{n+1})\right)\right) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha(\mathfrak{M}_{n-1}\mathfrak{M}_{n+1}) \dot{-} \alpha(\mathfrak{M}_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n-1}\mathfrak{M}_{n+1}) - \alpha(\mathfrak{M}_{n}^{2})\right)\right) \\ &= \alpha(\mathfrak{M}_{n-1}\mathfrak{M}_{n+1} - \mathfrak{M}_{n}^{2}) \\ &= \alpha(-2^{n-1}) = \dot{-}\dot{1} \dot{\times} \dot{2}^{\dot{n}\dot{-}s} \end{aligned}$$

(ii)
$$\begin{aligned} N\mathfrak{M}L_{n-1} \dot{\times} N\mathfrak{M}L_{n+1} \dot{-} N\mathfrak{M}L_{n}^{\dot{2}} &= \alpha(\mathfrak{M}L_{n-1}) \dot{\times} \alpha(\mathfrak{M}L_{n+1}) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \\ &= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n-1}) \times \alpha(\mathfrak{M}L_{n+1})\right)\right) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \\ &= \alpha(\mathfrak{M}L_{n-1}\mathfrak{M}L_{n+1}) \dot{-} \alpha(\mathfrak{M}L_{n}^{2}) \end{aligned}$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n-1}\mathfrak{M}L_{n+1}) - \alpha(\mathfrak{M}L_n^2)\right)\right)$$
$$= \alpha(\mathfrak{M}L_{n-1}\mathfrak{M}L_{n+1} - \mathfrak{M}L_n^2)$$
$$= \alpha(2^{n-1}) = \dot{2}^{\dot{n}\dot{-}\dot{1}}$$

Theorem 2.7 (d'Ocagne's identity)

(i) $N\mathfrak{M}_{n+1} \dot{\times} N\mathfrak{M}_m \dot{-} N\mathfrak{M}_n \dot{\times} N\mathfrak{M}_{m+1} = \dot{2}^{\dot{m}} \dot{-} \dot{2}^{\dot{n}}$

(ii) $N\mathfrak{M}L_{n+1} \dot{\times} N\mathfrak{M}L_m \dot{-} N\mathfrak{M}L_n \dot{\times} N\mathfrak{M}L_{m+1} = \dot{2}^{\dot{n}} \dot{-} \dot{2}^{\dot{m}}$

Proof:

(i) $N\mathfrak{M}_{n+1} \dot{\times} N\mathfrak{M}_m \dot{-} N\mathfrak{M}_n \dot{\times} N\mathfrak{M}_{m+1} = \alpha(\mathfrak{M}_{n+1}) \dot{\times} \alpha(\mathfrak{M}_m) \dot{-} \alpha(\mathfrak{M}_n) \dot{\times} \alpha(\mathfrak{M}_{m+1})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n+1}) \times \alpha(\mathfrak{M}_m)\right)\right) \dot{-} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_n) \times \alpha(\mathfrak{M}_{m+1})\right)\right)$$

$$= \alpha(\mathfrak{M}_{n+1}\mathfrak{M}_m) \dot{-} \alpha(\mathfrak{M}_n\mathfrak{M}_{m+1})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{n+1}\mathfrak{M}_m) - \alpha(\mathfrak{M}_n\mathfrak{M}_{m+1})\right)\right)$$

$$= \alpha(\mathfrak{M}_{n+1}\mathfrak{M}_m - \mathfrak{M}_n\mathfrak{M}_{m+1})$$

$$= \alpha(2^m - 2^n) = \dot{2}^{\dot{m}} \dot{-} \dot{2}^{\dot{n}}$$

(ii) $N\mathfrak{M}L_{n+1} \dot{\times} N\mathfrak{M}L_m \dot{-} N\mathfrak{M}L_n \dot{\times} N\mathfrak{M}L_{m+1} =$
$\alpha(\mathfrak{M}L_{n+1}) \dot{\times} \alpha(\mathfrak{M}L_m) \dot{-} \alpha(\mathfrak{M}L_n) \dot{\times} \alpha(\mathfrak{M}L_{m+1})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n+1}) \times \alpha(\mathfrak{M}L_m)\right)\right) \dot{-} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_n) \times \alpha(\mathfrak{M}L_{m+1})\right)\right)$$

$$= \alpha(\mathfrak{M}L_{n+1}\mathfrak{M}L_m) \dot{-} \alpha(\mathfrak{M}L_n\mathfrak{M}L_{m+1})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}L_{n+1}\mathfrak{M}L_m) - \alpha(\mathfrak{M}L_n\mathfrak{M}L_{m+1})\right)\right)$$

$$= \alpha(\mathfrak{M}L_{n+1}\mathfrak{M}L_m - \mathfrak{M}L_n\mathfrak{M}L_{m+1}) = \alpha(2^n - 2^m)$$

$$= \dot{2}^{\dot{n}} \dot{-} \dot{2}^{\dot{m}}$$

Theorem 2.8 (Honsberger Identity)

(i) $N\mathfrak{M}_{k-1} \dot{\times} N\mathfrak{M}_n \dot{+} N\mathfrak{M}_k \dot{\times} N\mathfrak{M}_{n+1} = \dot{5} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{k}\dot{-}\dot{1}} \dot{-} \dot{3} \dot{\times} \dot{2}^{\dot{n}} \dot{-} \dot{3} \dot{\times} \dot{2}^{\dot{k}\dot{-}\dot{1}} \dot{+} \dot{2}$

(ii) $N\mathfrak{M}L_{k-1} \dot{\times} N\mathfrak{M}L_n \dot{+} N\mathfrak{M}L_k \dot{\times} N\mathfrak{M}L_{n+1} = \dot{5} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{k}\dot{-}\dot{1}} \dot{+} \dot{3} \dot{\times} \dot{2}^{\dot{n}} \dot{+} \dot{3} \dot{\times} \dot{2}^{\dot{k}\dot{-}\dot{1}} \dot{+} \dot{2}$

Proof:

(i) $N\mathfrak{M}_{k-1} \dot{\times} N\mathfrak{M}_n \dot{+} N\mathfrak{M}_k \dot{\times} N\mathfrak{M}_{n+1} = \alpha(\mathfrak{M}_{k-1}) \dot{\times} \alpha(\mathfrak{M}_n) \dot{+} \alpha(\mathfrak{M}_k) \dot{\times} \alpha(\mathfrak{M}_{n+1})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}_{k-1}) \times \alpha(\mathfrak{m}_{n})\right)\right) \dot{+} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}_{k}) \times \alpha(\mathfrak{m}_{n+1})\right)\right)$$

$$= \alpha(\mathfrak{m}_{k-1}\mathfrak{m}_{n}) \dot{+} \alpha(\mathfrak{m}_{k}\mathfrak{m}_{n+1})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}_{k-1}\mathfrak{m}_{n}) + \alpha(\mathfrak{m}_{k}\mathfrak{m}_{n+1})\right)\right)$$

$$= \alpha(\mathfrak{m}_{k-1}\mathfrak{m}_{n} - \mathfrak{m}_{k}\mathfrak{m}_{n+1})$$

$$= \alpha(5.2^{n+k-1} - 3.2^{n} - 3.2^{k-1} + 2)$$

$$= \dot{5} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{k}\dot{-}\dot{1}} \dot{-} \dot{3} \dot{\times} \dot{2}^{\dot{n}} \dot{-} \dot{3} \dot{\times} \dot{2}^{\dot{k}\dot{-}\dot{1}} \dot{+} \dot{2}$$

(ii) $N\mathfrak{m}L_{k-1} \dot{\times} N\mathfrak{m}L_{n} \dot{+} N\mathfrak{m}L_{k} \dot{\times} N\mathfrak{m}L_{n+1} =$
$\alpha(\mathfrak{m}L_{k-1}) \dot{\times} \alpha(\mathfrak{m}L_{n}) \dot{+} \alpha(\mathfrak{m}L_{k}) \dot{\times} \alpha(\mathfrak{m}L_{n+1})$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}L_{k-1}) \times \alpha(\mathfrak{m}L_{n})\right)\right) \dot{+} \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}L_{k}) \times \alpha(\mathfrak{m}L_{n+1})\right)\right)$$

$$= \alpha(\mathfrak{m}L_{k-1}\mathfrak{m}L_{n}) \dot{+} \alpha(\mathfrak{m}L_{k}\mathfrak{m}L_{n+1})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{m}L_{k-1}\mathfrak{m}L_{n}) + \alpha(\mathfrak{m}L_{k}\mathfrak{m}L_{n+1})\right)\right)$$

$$= \alpha(\mathfrak{m}L_{k-1}\mathfrak{m}L_{n} + \mathfrak{m}L_{k}\mathfrak{m}L_{n+1})$$

$$= \alpha(5.2^{n+k-1} + 3.2^{n} + 3.2^{k-1} + 2)$$

$$= \dot{5} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{k}\dot{-}\dot{1}} \dot{+} \dot{3} \dot{\times} \dot{2}^{\dot{n}} \dot{+} \dot{3} \dot{\times} \dot{2}^{\dot{k}\dot{-}\dot{1}} \dot{+} \dot{2}$$

Theorem 2.9

(i) $N\mathfrak{m}_{n} = N\mathfrak{m}L_{n} \dot{-} \dot{2}$
(ii) $\alpha \sum_{i=0}^{n} N\mathfrak{m}_{i} = N\mathfrak{m}_{n+1} \dot{-} (\dot{n} \dot{+} \dot{1})$
(iii) $\alpha \sum_{i=0}^{n} N\mathfrak{m}L_{i} = N\mathfrak{m}L_{n+1} \dot{+} (\dot{n} \dot{-} \dot{1})$
(iv) $\alpha \sum_{i=0}^{n} (N\mathfrak{m}_{i} \dot{\times} N\mathfrak{m}L_{i}) = (\dot{n} \dot{+} \dot{1}) \dot{\times} N\mathfrak{m}_{n}$

Proof:

(i) $N\mathfrak{m}_{n} = \alpha(\mathfrak{m}_{n}) = \alpha(\mathfrak{m}_{n+1} - 2) = N\mathfrak{m}L_{n} \dot{-} \dot{2}$
(ii) $\alpha \sum_{i=0}^{n} N\mathfrak{m}_{i} = \alpha(\sum_{i=0}^{n} \mathfrak{m}_{i})$

$$= \alpha(\mathfrak{m}_{n+1} - (n+1)) = N\mathfrak{m}_{n+1} \dot{-} (\dot{n} \dot{+} \dot{1})$$

(iii) $\alpha \sum_{i=0}^{n} N\mathfrak{m}L_{i} = \alpha(\sum_{i=0}^{n} \mathfrak{m}L_{i})$

$$= \alpha(\mathfrak{m}L_{n+1} + (n-1)) = N\mathfrak{m}L_{n+1} \dot{+} (\dot{n} \dot{-} \dot{1})$$

(iv) $\alpha \sum_{i=0}^{n}(\mathrm{N}\mathfrak{m}_{\mathrm{i}} \dot{\times} \mathrm{N}\mathfrak{m}\mathrm{L}_{\mathrm{i}}) = \alpha(\sum_{i=0}^{n}(\mathrm{N}\mathfrak{m}_{\mathrm{i}}\mathrm{N}\mathfrak{m}\mathrm{L}_{\mathrm{i}}))$
$$= \alpha\big((n+1)\mathfrak{m}_{\mathrm{n}}\big) = \big(\dot{\mathrm{n}}\dot{+}\dot{1}\big) \dot{\times} \mathrm{N}\mathfrak{m}_{\mathrm{n}}$$

Theorem 2.10 Identities among Non-Newtonian Mersenne and Mersenne-Lucas sequences

$$\mathrm{N}\mathfrak{m}_{2n} = \mathrm{N}\mathfrak{m}_{n} \dot{\times} \mathrm{N}\mathfrak{m}L_{n}$$

$$\mathrm{N}\mathfrak{m}_{2n+1} = \mathrm{N}\mathfrak{m}_{n} \dot{\times} \mathrm{N}\mathfrak{m}L_{n+1} \dot{+} \dot{2}^{\dot{n}}$$

$$\mathrm{N}\mathfrak{m}_{2n+1} = \frac{\dot{1}}{\dot{2}}\alpha \dot{\times} (\mathrm{N}\mathfrak{m}_{n} \dot{\times} \mathrm{N}\mathfrak{m}L_{n+1} \dot{+} \mathrm{N}\mathfrak{m}L_{n} \dot{\times} \mathrm{N}\mathfrak{m}_{n+1})$$

$$\mathrm{N}\mathfrak{m}L_{2n+1} = \frac{\dot{1}}{\dot{2}}\alpha \dot{\times} (\mathrm{N}\mathfrak{m}_{n} \dot{\times} \mathrm{N}\mathfrak{m}_{n+1} \dot{+} \mathrm{N}\mathfrak{m}L_{n} \dot{\times} \mathrm{N}\mathfrak{m}L_{n+1})$$

$$\mathrm{N}\mathfrak{m}L_{2n} = \mathrm{N}\mathfrak{m}_{n} \dot{\times} \mathrm{N}\mathfrak{m}L_{n} \dot{+} \dot{2}$$

Theorem 2.11

$\mathrm{N}\mathfrak{m}_{2n} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n} = \mathrm{N}\mathfrak{m}_{4n}$, where $n \geq 1$.

Proof

$$\mathrm{N}\mathfrak{m}_{2n} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n} = \alpha(\mathfrak{m}_{2n}) \dot{\times} \alpha(\mathfrak{m}L_{2n})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{m}_{2n}) \times \alpha(\mathfrak{m}L_{2n})\big)\right)$$

$$= \alpha(\mathfrak{m}_{2n}\mathfrak{m}L_{2n}) = \alpha(\mathfrak{m}_{4n}) = \mathrm{N}\mathfrak{m}_{4n}$$

Theorem 2.12

$\mathrm{N}\mathfrak{m}_{2n} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n+2} = \mathrm{N}\mathfrak{m}_{4n+2} \dot{-} \big(\dot{2}^{\dot{2}\dot{n}}\big) \dot{\times} 3$, where $n \geq 1$.

Proof

$$\mathrm{N}\mathfrak{m}_{2n} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n+2} = \alpha(\mathfrak{m}_{2n}) \dot{\times} \alpha(\mathfrak{m}L_{2n+2})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{m}_{2n}) \times \alpha(\mathfrak{m}L_{2n+2})\big)\right)$$

$$= \alpha(\mathfrak{M}_{2n}\mathfrak{M}L_{2n+2}) = \alpha\big(\mathfrak{M}_{4n+2} - 3(2^{2n})\big) = \mathrm{N}\mathfrak{M}_{4n+2} \dot{-} \big(\dot{2}^{\dot{2}\dot{n}}\big) \dot{\times} 3$$

Theorem 2.13

$\mathrm{N}\mathfrak{M}_{2n} \dot{\times} \mathrm{N}\mathfrak{M}L_{2n+1} = \mathrm{N}\mathfrak{M}_{4n+1} \dot{-} \big(\dot{2}^{\dot{2}\dot{n}}\big)$, where $n \geq 1$.

Proof

$$\mathrm{N}\mathfrak{M}_{2n}\mathrm{N}\mathfrak{M}L_{2n+1} = \alpha(\mathfrak{M}_{2n}) \dot{\times} \alpha(\mathfrak{M}L_{2n+1})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{M}_{2n}) \times \alpha(\mathfrak{M}L_{2n+1})\big)\right)$$

$$= \alpha(\mathfrak{M}_{2n}\mathfrak{M}L_{2n+1})$$

$$= \alpha\big(\mathfrak{M}_{4n+1} - (2^{2n})\big)$$

$$= \mathrm{N}\mathfrak{M}_{4n+1} \dot{-} \big(\dot{2}^{\dot{2}\dot{n}}\big)$$

Theorem 2.14

$\mathrm{N}\mathfrak{M}_{2n} \dot{\times} \mathrm{N}\mathfrak{M}L_{2n+3} = \mathrm{N}\mathfrak{M}_{4n+3} \dot{-} \dot{7} \dot{\times} \big(\dot{2}^{\dot{2}\dot{n}}\big)$, where $n \geq 1$.

Proof

$$\mathrm{N}\mathfrak{M}_{2n} \dot{\times} \mathrm{N}\mathfrak{M}L_{2n+3} = \alpha(\mathfrak{M}_{2n}) \dot{\times} \alpha(\mathfrak{M}L_{2n+3})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{M}_{2n}) \times \alpha(\mathfrak{M}L_{2n+3})\big)\right)$$

$$= \alpha(\mathfrak{M}_{2n}\mathfrak{M}L_{2n+3})$$

$$= \alpha\big(\mathfrak{M}_{4n+3} - 7(2^{2n})\big)$$

$$= \mathrm{N}\mathfrak{M}_{4n+3} \dot{-} \dot{7} \dot{\times} \big(\dot{2}^{\dot{2}\dot{n}}\big)$$

Theorem 6.15

$\mathrm{N}\mathfrak{M}_{2n-1} \dot{\times} \mathrm{N}\mathfrak{M}L_{2n+1} = \mathrm{N}\mathfrak{M}_{4n} \dot{-} \dot{3} \dot{\times} \big(\dot{2}^{\dot{2}\dot{n} \dot{-} \dot{1}}\big)$, where $n \geq 1$.

Proof

$$\mathrm{N}\mathfrak{m}_{2n-1}\mathrm{N}\mathfrak{m}L_{2n+1} = \alpha(\mathfrak{m}_{2n-1}) \dot{\times} \alpha(\mathfrak{m}L_{2n+1})$$

$$= \alpha\left(\alpha^{-1}(\alpha(\mathfrak{m}_{2n-1}) \times \alpha(\mathfrak{m}L_{2n+1}))\right)$$

$$= \alpha(\mathfrak{m}_{2n-1}\mathfrak{m}L_{2n+1})$$

$$= \alpha\left(\mathfrak{m}_{4n} - 3(2^{2n-1})\right)$$

$$= \mathrm{N}\mathfrak{m}_{4n} \dot{-} \dot{3} \dot{\times} \left(\dot{2}^{\dot{2}\dot{n}\dot{-}\dot{1}}\right)$$

Theorem 2.16

$$\mathrm{N}\mathfrak{m}_{2n+1} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n} = \mathrm{N}\mathfrak{m}_{4n+1} \dot{+} \left(\dot{2}^{\dot{2}\dot{n}}\right), \text{ where } n \geq 1.$$

Proof

$$\mathrm{N}\mathfrak{m}_{2n+1} \dot{\times} \mathrm{N}\mathfrak{m}L_{2n} = \alpha(\mathfrak{m}_{2n+1}) \dot{\times} \alpha(\mathfrak{m}L_{2n})$$

$$= \alpha\left(\alpha^{-1}(\alpha(\mathfrak{m}_{2n+1}) \times \alpha(\mathfrak{m}L_{2n}))\right)$$

$$= \alpha(\mathfrak{m}_{2n+1}\mathfrak{m}L_{2n})$$

$$= \alpha\left(\mathfrak{m}_{4n+1} + (2^{2n})\right)$$

$$= \mathrm{N}\mathfrak{m}_{4n+1} \dot{+} \left(\dot{2}^{\dot{2}\dot{n}}\right)$$

Theorem 2.17

$$\mathrm{N}\mathfrak{m}_{m} \dot{\times} \mathrm{N}\mathfrak{m}L_{n} = \mathrm{N}\mathfrak{m}_{m+n} \dot{-} \dot{2}^{\dot{m}} \dot{\times} \mathrm{N}\mathfrak{m}_{n-m}, \text{ where } n \geq 1,\ \ m \geq 0.$$

Proof

$$\mathrm{N}\mathfrak{m}_{m} \dot{\times} \mathrm{N}\mathfrak{m}L_{n} = \alpha(\mathfrak{m}_{m}) \dot{\times} \alpha(\mathfrak{m}L_{n})$$

$$= \alpha\left(\alpha^{-1}(\alpha(\mathfrak{m}_{m}) \times \alpha(\mathfrak{m}L_{n}))\right)$$

$$= \alpha(\mathfrak{m}_m \mathfrak{m}L_n) = \alpha(\mathfrak{m}_{m+n} - 2^m \mathfrak{m}_{n-m}) = N\mathfrak{m}_{m+n} \dot{-} \dot{2}^{\dot{m}} \dot{\times} N\mathfrak{m}_{n-m}$$

Theorem 2.18

$$N\mathfrak{m}_n \dot{\times} N\mathfrak{m}L_{2n+m} = N\mathfrak{m}_{3n+m} \dot{-} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{m}_{n+m}, \text{ where } n \geq 1,\ \ m \geq 0.$$

Proof

$$N\mathfrak{m}_n \dot{\times} N\mathfrak{m}L_{2n+m} = \alpha(\mathfrak{m}_n) \dot{\times} \alpha(\mathfrak{m}L_{2n+m})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{m}_n) \times \alpha(\mathfrak{m}L_{2n+m})\big)\right)$$

$$= \alpha(\mathfrak{m}_n \mathfrak{m}L_{2n+m}) \qquad = \alpha(\mathfrak{m}_{3n+m} - 2^n \mathfrak{m}_{n+m})$$

$$= N\mathfrak{m}_{3n+m} \dot{-} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{m}_{n+m}$$

Theorem 2.19

$$N\mathfrak{m}_{2n+m} \dot{\times} N\mathfrak{m}L_n = N\mathfrak{m}_{3n+m} \dot{+} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{m}_{n+m}, \text{ where } n \geq 1,\ \ m \geq 0.$$

Proof

$$N\mathfrak{m}_{2n+m} \dot{\times} N\mathfrak{m}L_n = \alpha(\mathfrak{m}_{2n+m}) \dot{\times} \alpha(\mathfrak{m}L_n)$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{m}_{2n+m}) \times \alpha(\mathfrak{m}L_n)\big)\right)$$

$$= \alpha(\mathfrak{m}_{2n+m} \mathfrak{m}L_n)$$

$$= \alpha(\mathfrak{m}_{3n+m} + 2^n \mathfrak{m}_{n+m}) = N\mathfrak{m}_{3n+m} \dot{+} \dot{2}^{\dot{n}} \dot{\times} N\mathfrak{m}_{n+m}$$

Theorem 2.20

$$N\mathfrak{m}_{2n} \dot{\times} N\mathfrak{m}L_{2n+m} = N\mathfrak{m}_{4n+m} \dot{-} \left(\dot{2}^{\dot{2}\dot{n}}\right) \dot{\times} N\mathfrak{m}_m, \text{ where } n \geq 1,\ \ m \geq 0.$$

Proof

$$N\mathfrak{m}_{2n} \dot{\times} N\mathfrak{m}L_{2n+m} = \alpha(\mathfrak{m}_{2n}) \dot{\times} \alpha(\mathfrak{m}L_{2n+m})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{m}_{2n}) \times \alpha(\mathfrak{m}L_{2n+m})\big)\right)$$

$$= \alpha(\mathfrak{M}_{2n}\mathfrak{M}L_{2n+m}) \quad = \alpha(\mathfrak{M}_{4n+m} - (2^{2n})\mathfrak{M}_{m})$$

$$= N\mathfrak{M}_{4n+m} \dot{-} (\dot{2}^{\dot{2}\dot{n}}) \dot{\times} N\mathfrak{M}_{m}$$

Theorem 2.21

$N\mathfrak{M}_{2n+m} \dot{\times} N\mathfrak{M}L_{2n} = N\mathfrak{M}_{4n+m} \dot{+} \dot{2}^{\dot{2}\dot{n}} \dot{\times} N\mathfrak{M}_{m}$, where $n \geq 1,\ \ m \geq 0.$

Proof

$$N\mathfrak{M}_{2n+m} \dot{\times} N\mathfrak{M}L_{2n} = \alpha(\mathfrak{M}_{2n+m}) \dot{\times} \alpha(\mathfrak{M}L_{2n})$$

$$= \alpha\left(\alpha^{-1}\left(\alpha(\mathfrak{M}_{2n+m}) \times \alpha(\mathfrak{M}L_{2n})\right)\right)$$

$$= \alpha(\mathfrak{M}_{2n+m}\mathfrak{M}L_{2n})$$

$$= \alpha(\mathfrak{M}_{4n+m} + 2^{2n}\mathfrak{M}_{m}) = N\mathfrak{M}_{4n+m} \dot{+} \dot{2}^{\dot{2}\dot{n}} \dot{\times} N\mathfrak{M}_{m}$$

Theorem 2.22

The sums of their consecutive terms are

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}_{i} = \frac{N\mathfrak{M}_{n+2} \dot{-} \dot{2}\dot{n} \dot{-} \dot{3}}{\dot{2}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}_{2i} = \frac{N\mathfrak{M}_{2n+2} \dot{-} \dot{3}\dot{n} \dot{-} \dot{3}}{\dot{3}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}_{i+1} = \frac{N\mathfrak{M}_{n+3} \dot{-} \dot{2}\dot{n} \dot{-} \dot{7}}{\dot{2}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}_{2i+1} = \frac{\dot{2}N\mathfrak{M}_{2n+2} \dot{-} \dot{3}\dot{n} \dot{-} \dot{6}}{\dot{3}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}L_{i} = \frac{N\mathfrak{M}L_{n+2} \dot{+} \dot{2}\dot{n} \dot{-} \dot{5}}{\dot{2}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}L_{i+1} = \frac{N\mathfrak{M}L_{n+3} \dot{+} \dot{2}\dot{n} \dot{-} \dot{9}}{\dot{2}} \alpha$$

$$\alpha \sum_{i=1}^{n} N\mathfrak{M}L_{2i} = \frac{N\mathfrak{M}L_{2n+2} \dot{+} \dot{3}\dot{n} \dot{-} \dot{5}}{\dot{3}} \alpha$$

$$\alpha \sum_{i=1}^{n} \text{N}\mathfrak{M}L_{2i+1} = \frac{\text{N}\mathfrak{M}L_{2n+3} \dot{+} \dot{3}\dot{n} \dot{-} 9}{\dot{3}} \alpha$$

Theorem 6.23

Let $\text{N}\mathfrak{M}_n$ be the n^{th} Non-Newtonian Mersenne number. Let n and k be arbitrary positive integers then the following equality is satisfied.

$$\text{N}\mathfrak{M}_n \dot{\times} \text{N}\mathfrak{M}_{n+k} = \text{N}\mathfrak{M}_{2n+k} \dot{-} \text{N}\mathfrak{M}_{n+k} \dot{-} \text{N}\mathfrak{M}_n$$

Proof

$$\text{N}\mathfrak{M}_n \dot{\times} \text{N}\mathfrak{M}_{n+k} = \alpha(\mathfrak{M}_n) \dot{\times} \alpha(\mathfrak{M}L_{n+k})$$

$$= \alpha\left(\alpha^{-1}\big(\alpha(\mathfrak{M}_n) \times \alpha(\mathfrak{M}L_{n+k})\big)\right)$$

$$= \alpha(\mathfrak{M}_n \mathfrak{M}L_{n+k}) = \alpha(\mathfrak{M}_{2n+k} - \mathfrak{M}_{n+k} - \mathfrak{M}_n)$$

$$= \text{N}\mathfrak{M}_{2n+k} \dot{-} \text{N}\mathfrak{M}_{n+k} \dot{-} \text{N}\mathfrak{M}_n$$

Taking $k = 0$ and $k = 1$, we get

$$\text{N}\mathfrak{M}_n^2 = \text{N}\mathfrak{M}_{2n} \dot{-} \dot{2} \dot{\times} \text{N}\mathfrak{M}_n$$

$$\text{N}\mathfrak{M}_n \dot{\times} \text{N}\mathfrak{M}_{n+1} = \text{N}\mathfrak{M}_{2n+1} \dot{-} \text{N}\mathfrak{M}_{n+1} \dot{-} \text{N}\mathfrak{M}_n$$

3. Conclusion:

In this paper, we have introduced a new class of sequences, named the non-Newtonian Mersenne and non-Newtonian Mersenne-Lucas numbers. . The Binet's formulas and generating functions for these sequences are obtained. Some interesting identities about these sequences are also presented.

References:

1. Dişkaya, O. (2024). On the Non-Newtonian Padovan and Non-Newtonian Perrin Numbers. Sinop Üniversitesi Fen Bilimleri Dergisi, 9(2), 502-515.
2. Duyar C and M. Erdo˘gan, On non-Newtonian real number series, IOSR Journal of Mathematics 12(6), (2016) 34–48.

3. Ilknur Yes¸ilyurta , Nilay De˘girmen, “Non-Newtonian Jacobsthal and Jacobsthal-Lucas numbers: A new look”, Filomat 39:4 (2025), 1093–1109.
4. M. Özdemir and A. A. Ergin, “Rotations With Unit Timelike Quaternions in Minkowski 3-Space,” Journal of Geometry and Physics **56**, no. 2 (2006): 322–336.
5. Maheswari D and Devibala S, Unrestricted Mersenne and Mersenne-Lucas Hybrid Quaternions, Jnanabha, 54(2) (2024), 113-119.
6. Malini Devi B., Devibala S., A view on Mersenne and Mersenne-Lucas numbers, GIS SCIENCE JOURNAL, Vol. 8, Issue 8, Pp 784-789, August 2021.
7. Malini Devi B., Devibala S., Some sums formulae for products of terms of Mersenne and Mersenne-Lucas Numbers, Adalya Journal, Vol. 10, Issue 8, Pp 74-79, August 2021.
8. Malini Devi B., Devibala S., The Exponential Generating functions of Mersenne and Mersenne-Lucas Identities, Turkish Journal of Computer and Mathematics Education, Vol. 12, No. 14, Pp 739-747, 2021.
9. N. De˘girmen, C. Duyar, A new perspective on Fibonacci and Lucas numbers, Filomat 37 (28)(2023) 9561–9574.
10. Paula Catarino, Helena Campos & Paulo Vasco (2016). On the Mersenne sequences, Annales Mathematicae et informaticae, 46, 37-53.